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STABILITY ANALYSIS OF STRUCTURES BY A REDUCED SYSTEM OF GENERALIZED COORDINATES

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Abstract—A method of analysis using a reduced system of generalized coordinates for the static linear bifurcation theory of stability is presented. Finite elements (displacement models) are used to idealize the structure. The advantage of the present method is a large reduction in the size of the eigenvalue problem to be solved without sacrificing the capability of using a large number of degrees of freedom for mathematical modeling. The efficiency and accuracy are illustrated by numerical examples for various types of planar structures. Although only examples using one-dimensional type elements are offered, this procedure is obviously equally applicable for other structures such as plates and shells.

1. INTRODUCTION

THE foundations of the linear theory of static stability of structures are well established. Formulation of the governing equations by equilibrium methods results in an eigenvalue problem whose solution consists of a denumerable set of eigenvalues and a corresponding set of eigenfunctions. These are related to the critical loads and stable equilibrium configurations. Of course, only the lowest eigenvalue is of practical interest. Alternatively, energy methods have been employed. In this case, an approximate value of the critical load is obtained with the use of one or more assumed coordinate functions. Monographs such as Timoshenko and Gere [1], Bleich [2] and Ziegler [3] give illustrations of these principles to various structures.

Recently, finite element methods (displacement models) have been proposed for the solution to stability problems; see, for example, Hartz [4], Kapur and Hartz [5], Navaratna *et al.* [6] and Anderson *et al.* [7]. In this method, a discrete number of coordinates is chosen as the primary variables for the formulation of the problem. A system of algebraic equations of the form

$$[K]\{U\}+\lambda[K_G]\{U\} = 0 \tag{1}$$

is obtained, where $[K]$ is the stiffness, $[K_G]$ the geometrical stiffness or initial stress matrix and $\{U\}$ an ordered set of nodal point displacements. The coefficient λ is related to the critical load configuration. Methods of solution to equation (1) fall into two general categories: iteration and direct solution (Jacobi, Householder, etc.). In the former, a value for λ is assumed and the vanishing of the determinant of $\{[K]+\lambda[K_G]\}$ is checked. This procedure is repeated until the correct value is found. In the direct method, a solution of the form

$$\{U\} = [\phi]\{X\} \tag{2}$$

is sought with the properties

$$[\phi]^T[K_G][\phi] = [I] \tag{3}$$

$$[\phi]^T[K][\phi] = -[\lambda_n]. \tag{4}$$

The modal columns in $[\phi]$ represent the eigenvectors and $\{X\}$ are the "normal" coordinates of the system.

For the solution to the stability of complex structures and structural systems, an accurate mathematical model is necessary, requiring an extensive number of nodal points. This will result in matrices $[K]$ and $[K_G]$ of considerable size. Consequently, for a large number of coordinates, the computational effort by the direct method becomes inordinately large. In this paper, a method is proposed which transforms the generalized coordinates of the original system to a reduced system of generalized coordinates. Then, the direct solution of the problem is effected in the reduced generalized coordinates. By this method, the capability of using a large number of nodes to model the structure is retained and the advantage of solving an eigenvalue problem of very limited size is achieved. This method is akin to the Rayleigh–Ritz procedure. Examples are given to illustrate the accuracy, versatility and efficiency of the method. Although only examples of one-dimensional members are offered, this procedure obviously is equally applicable for other structures such as plates and shells.

2. PROBLEM STATEMENT

Consider a planar structure with arbitrary geometry and variable linear elastic properties as shown in Fig. 1, which is acted upon by a system of forces $F_1, F_2, \ldots, F_i, \ldots, F_n$. Let a parameter λ be introduced as a multiplier of every load F_i. This parameter has the significance of the ratio of a critical loading configuration to the present loading configuration. We are primarily concerned with determining the least value of λ, which is associated with the loading configuration at the inception of instability when the structure will assume an alternate planar equilibrium position that is stable.

Stability analysis by the finite element method requires that the structure be subdivided into a system of elements which are connected at a discrete number of nodes. Each element is assumed to have prismatic properties: l, A, I, E which are, respectively,

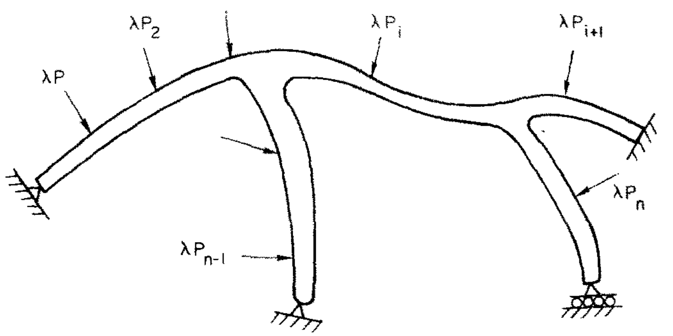


FIG. 1. Elastic structure and external loads.

its length, cross-sectional area, moment of inertia with respect to an axis normal to the plane and modulus of elasticity. The bases of the method are matrix expressions for the stiffness and geometrical stiffness, which are relationships between the nodal point forces and displacements of an element. The stiffness is a function of the mechanical and geometrical properties of the element, whereas the geometrical stiffness depends primarily on the axial force P_j, which results as the structure is deformed from its natural state by the system of loads F_i. These relationships are known and we will adopt the forms used by Martin [8], who has traced the history of their development. The element force–deformation relationship in the deformed state is

$$\{f_j\} = \Big[[k_j] + \lambda[k_{gj}]\Big]\{\delta_j\} \tag{5}$$

where $\{f_j\}$ and $\{\delta_j\}$ are force and displacement vectors referred to a global coordinate system, see Fig. 2.

$$\{f_j\}^T = \{X_b, Y_b, M_b, X_f, Y_f, M_f\} \tag{6a}$$

$$\{\delta_j\}^T = \{u_b, v_b, \beta_b, u_f, v_f, \beta_f\}. \tag{6b}$$

Elements of the stiffness $[k_j]$ and geometrical stiffness $[k_{gj}]$ matrices are:

$$[k_j] = \frac{EI}{l^3}\begin{bmatrix} 12s^2+\mu c^2 & -(12-\mu)sc & -6sl & -12s^2-\mu c^2 & (12-\mu)sc & -6sl \\ & 12c^2+\mu s^2 & 6cl & (12-\mu)sc & -12c^2-\mu s^2 & 6cl \\ & & 4l^2 & 6sl & -6cl & 2l^2 \\ & & & 12s^2+\mu c^2 & -(12-\mu)sc & 6sl \\ & \text{Symmetric} & & & 12c^2+\mu s^2 & -6cl \\ & & & & & 4l^2 \end{bmatrix} \tag{7}$$

$$[k_{gj}] = \frac{P_j}{30l}\begin{bmatrix} 36s^2 & -36sc & -3sl & -36s^2 & 36sc & -3sl \\ & 36c^2 & 3cl & 36sc & -36c^2 & 3cl \\ & & 4l^2 & 3sl & -3cl & -l^2 \\ & \text{Symmetric} & & 36s^2 & -36sc & 3sl \\ & & & & 36c^2 & -3cl \\ & & & & & 4l^2 \end{bmatrix} \tag{8}$$

where P_j is the axial force (considered positive for tension) and α is the inclination of the member with the x-axis, see Fig. 2. The remaining notation used in equations (7) and (8) is defined as:

$$\mu = Al^2/I,\ c = \cos\alpha,\ s = \sin\alpha. \tag{9}$$

In the absence of initial stress (the natural state), note that the force–deformation relation is that given by equation (5) with $[k_{gj}] = 0$.

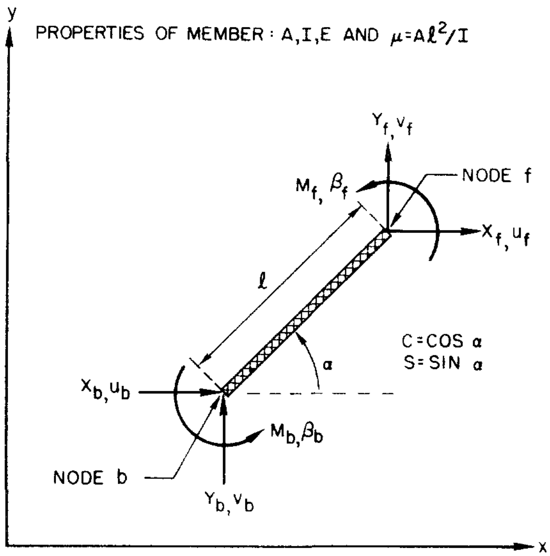


FIG. 2. Element geometry.

Specific steps in stability analysis are as follows:

(1) The element stiffness relationships $[k_j]$ are evaluated and combined in accordance with equilibrium and displacement compatibility requirements at each node to yield:

$$[K]\{U\} = \lambda\{F\} \tag{10}$$

where $\lambda\{F\}$ is the external load vector, $[K]$ is the stiffness of the assembled elements and $\{U\}$ is the ordered set of nodal displacements. The solution to equation (10) furnishes an equilibrium configuration described by $\{U\}$, which will be called state (e). From $\{U\}$, the element axial forces P_j can be computed.

(2) To investigate the stability of state (e), let an admissible neighboring equilibrium (p) be introduced, which is described by a displacement field $\{U^*\}$ measured relative to equilibrium state (e). In order to ascertain the existence of a nontrivial $\{U^*\}$, the stiffness $[K]$ is reconstructed and $[K_G]$ is assembled from the element geometrical stiffnesses $[k_{gj}]$, using P_j, the known element axial forces. The second variation of the potential energy of internal forces $\frac{1}{2}\delta^2 V$, which is denoted by Q, between states (e) and (p) is given by:

$$Q = \tfrac{1}{2}\{U^*\}^T \Big[[K]+\lambda[K_G]\Big] \{U^*\}. \tag{11}$$

Trefftz's criterion [9] states that for Q to possess a nontrivial minimum, where its character changes from positive definite to positive semidefinite and signifies that the buckling load has been reached, the variational equation

$$\delta Q = 0 \tag{12}$$

must be satisfied. Equation (12) yields a system of linear algebraic equations for a set of nontrivial displacements.

$$\Big[[K]+\lambda[K_G]\Big] \{U^*\} = 0. \tag{13}$$

The solution to this eigenvalue problem gives the magnitudes of λ_i, which are the ratios of critical loading configurations to the loading configuration described by $\{F\}$.

3. ANALYSIS BY GENERALIZED COORDINATES

In order to retain the capability of using a large number of degrees of freedom to model a structure and at the same time achieve the advantage of solving an eigenvalue problem of limited size, the following procedure is suggested. Consider the following displacement transformation from the generalized coordinates to a reduced system of generalized coordinates:

$$\{U^*\} = [\Psi]\{h\} \tag{14}$$

where $\{h\}$ is a reduced generalized displacement vector of smaller size. The columns of $[\Psi]$ represent a set of independent displacement configurations satisfying geometric continuity and geometric boundary conditions. These column vectors must be capable of giving a relatively complete description (in a mathematical sense) of the buckled configurations and can be found by the solution of

$$[K][\Psi] = [H] \tag{15}$$

where each column of $[H]$ represents an independent static load pattern. The load patterns must be chosen such that the solution of equation (15) for $[\Psi]$ approximates the primary buckled configurations. A direct solution for $[\Psi]$ may be found at the same time as the solution for $\{U\}$ of state (e) [step (1) of the previous section].

The second variation of the strain energy, given by equation (11) and expressed in terms of generalized coordinates $\{h\}$, takes the form

$$Q = \tfrac{1}{2}\{h\}^T \left[[\bar{K}]+\lambda[\bar{K}_G]\right] \{h\} \tag{16}$$

where

$$[\bar{K}] = [\Psi]^T[K][\Psi] \tag{17}$$

$$[\bar{K}_G] = [\Psi]^T[K_G][\Psi]. \tag{18}$$

The sizes of $[\bar{K}]$ and $[\bar{K}_G]$ resulting from computations (17) and (18) are much smaller than their corresponding original counterparts. Applying Trefftz's criterion to equation (16) yields

$$\left[[\bar{K}]+\lambda[\bar{K}_G]\right] \{h\} = 0 \tag{19}$$

which represents the eigenvalue problem to be solved. The solution to equation (19) is

$$\{h\} = [\varphi']\{X\} \tag{20}$$

where $\{X\}$ is a set of "normal coordinates" and $[\varphi']$ are eigenvectors possessing the properties:

$$[\varphi']^T[\bar{K}_G][\varphi'] = [I] \tag{21}$$

$$[\varphi']^T[\bar{K}][\varphi'] = -[\lambda_n] \tag{22}$$

where λ_n are the eigenvalues of the problem. As the basis $[\varphi']$ spans a smaller generalized space, the matrix product

$$[\phi] = [\Psi][\varphi'] \tag{23}$$

represents a subset of eigenvectors of the structure in the physical coordinates associated with the eigenvalues λ_n. These eigenvectors correspond to the lowest eigenvalues of the system, but this can only be true if the lowest eigenvectors can be found from $[\Psi]$. For example, Shubinski *et al.* [10] have shown for vibration problems that $[\phi]$ are the lowest eigenvectors provided they could be extracted from the reduced set of generalized coordinates.

4. EXAMPLES

The following examples are offered to illustrate the accuracy and efficiency of the present method. A computer code prepared in FORTRAN IV for an IBM 360/75 model was employed for the analysis. Examples were divided into two general categories. In the first group, an attempt was made to check the accuracy of the present method. Static stability analyses of columns and frames were performed. For such simple geometrical configurations, no added advantage is derived in using the present method. These examples are offered only to develop confidence in the method. In the second group, consideration is given to examples in which many degrees of freedom are needed to properly model the geometrical configuration. The merit of the present method will evince itself in these examples.

The type of problems analyzed in the first group together with the number of elements chosen to represent each type of structure are summarized in Table 1. Section properties A and EI are constant throughout the structural system in all examples except for the stepped column. Also summarized there are the end conditions and loading functions used to generate the reduced system of generalized coordinates. A description of pertinent details for each example follows:

(i) *Uniform column.* Critical loads were calculated for a prismatic bar with four sets of end conditions. Complete results are given in Table 2. Note that for the hinged–hinged case, the independent loading functions chosen are proportional to the exact buckled shapes. Thus, all ten critical loads are in excellent agreement with theory. For other end conditions, the independent loads are not proportional to the buckled shapes. Nevertheless, very good agreement is obtained for all but the highest critical loads.

(ii) *Stepped cantilever column.* The lowest computed critical load was $1{\cdot}0557EI/L^2$ and the theoretical value given in [1, p. 115] was $1{\cdot}055EI/L^2$.

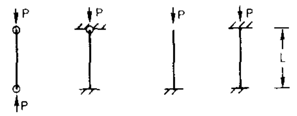
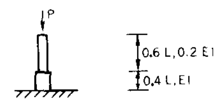
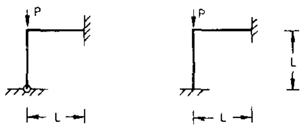
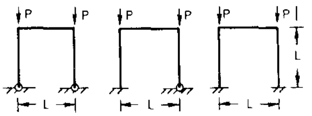
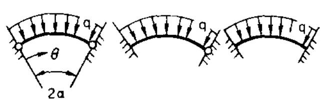
(iii) *Two bar frame.* For a frame with a hinged column and a fixed beam, the lowest computed critical load was $14{\cdot}66EI/L^2$, and for a fixed–fixed frame, the lowest computed critical load was $28{\cdot}40EI/L^2$. Theoretical values presented in [1, p. 65] were $14{\cdot}66EI/L^2$ and $28{\cdot}38EI/L^2$, respectively.

(iv) *Bent.* The bent analyzed was composed of three prismatic bars of equal length. Critical loads and buckled shapes were calculated for three sets of end conditions. The computed results are shown in Fig. 3 for the lowest symmetric and the lowest antisymmetric buckling loads for each set of end conditions. The results are in excellent agreement.

Hartz [4] also analyzed the bent by the direct solution of the equations of a finite element formulation, where he modeled the hinged–hinged case and fixed–fixed case as problems with three and two generalized coordinates, respectively. Using the present method with an equal number of generalized loads in each case, approximately the same results were obtained. No added advantage is realized in using the present method for simple geometries, as indicated previously.

The usefulness and practicality of the proposed method can be shown by the following examples.

TABLE 1. PROBLEM DESCRIPTIONS

Structure	Number of elements	Number of degrees of freedom	Types of buckling loads	Description of independent loads
Column	20	63		Transverse loads on column $q_n = \sin\frac{(2n-1)\pi x}{2L}$
Stepped column	20	63		for fixed-free end conditions, and $q_n = \sin\frac{n\pi x}{L}$ for all other cases, with
Two bar frame	40	123		$n = 1, 2, \ldots, 10$, and x, the running distance along the column
Bent	60	183		Transverse loads on columns $q_n = \sin\frac{n\pi x}{L}$, $n = 1, 2, \ldots, 5$ 5 symmetric and 5 antisymmetric cases
Shallow arch	40	123		Normal pressure loads $q_n = \sin\frac{n\pi\theta}{2\alpha}$ $n = 1, 2, \ldots, 10$

(v) *Shallow arch.* Buckling loads and buckled shapes were calculated for a uniformly loaded shallow circular arch using three sets of end conditions. The results are shown in Fig. 4 for three ratios of AR^2/I. Theoretical results used in the comparison were obtained considering the arch as inextensible. For values of extensional rigidity which are large in comparison with flexural rigidity, the computed values are in excellent agreement. It is interesting to note that for the present case the arch buckles in a symmetric shape when the ratio AR^2/I decreases.

(vi) *Deep arch.* Buckling load parameters for a deep arch with fixed supports on both ends were calculated and compared with the analytical solution of Wempner and Kesti [11]. The independent loads used are the same as those given for the shallow arch in Table 1. The case under comparison is that which Wempner and Kesti called constant direction theory. In addition, the direct solution of the eigenvalue problem using four elements was made. This problem as posed contained nine degrees of freedom. Therefore, only nine generalized loads were used in the present method in order to afford a valid comparison. The results are presented in Table 3. It can be seen that the present method yields much more accurate results than the four element model.

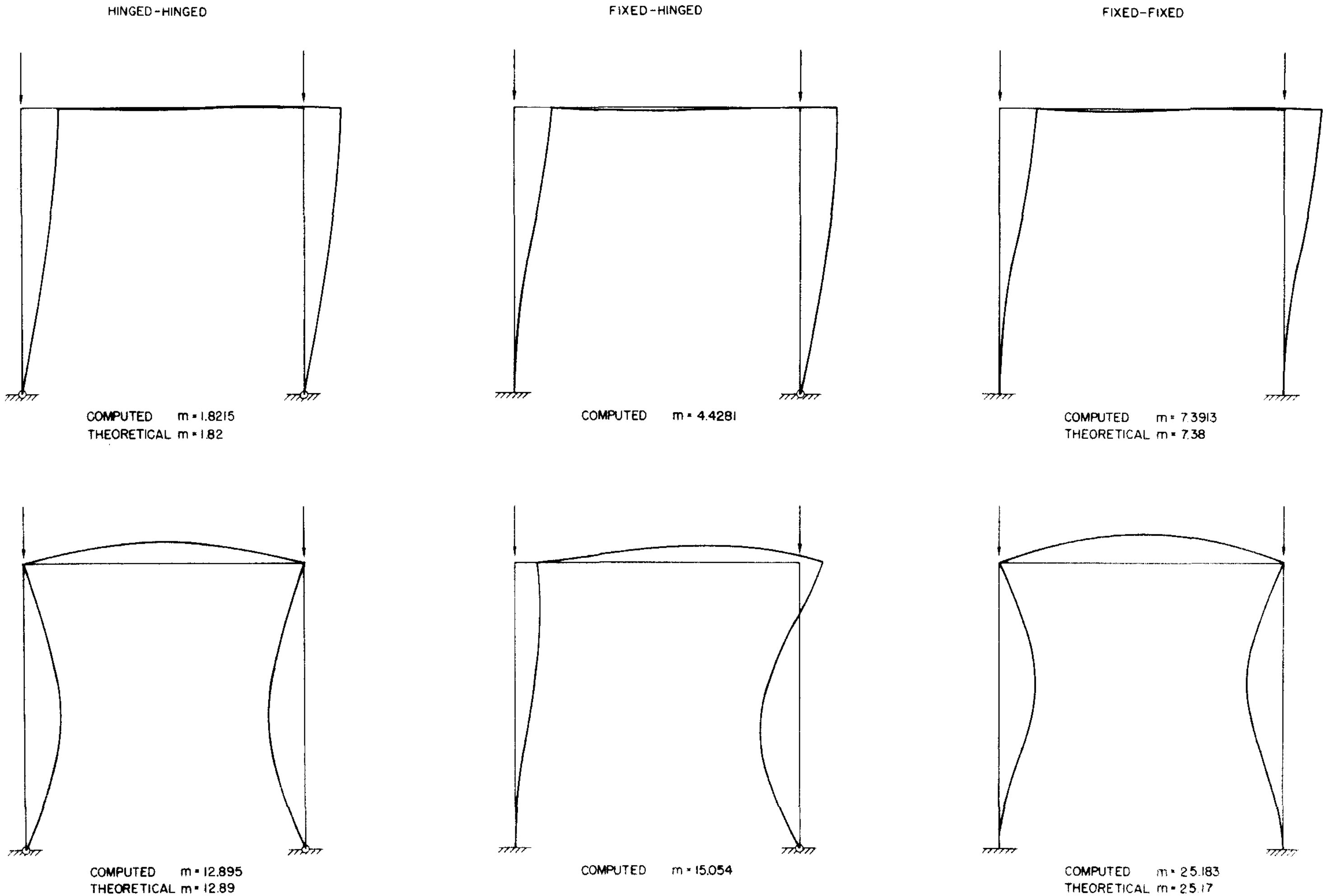


FIG. 3. Buckling loads and buckled shapes for bents, $P_{cr} = \dfrac{mEI}{L^2}$ (theoretical values from [2] pp. 253–254).

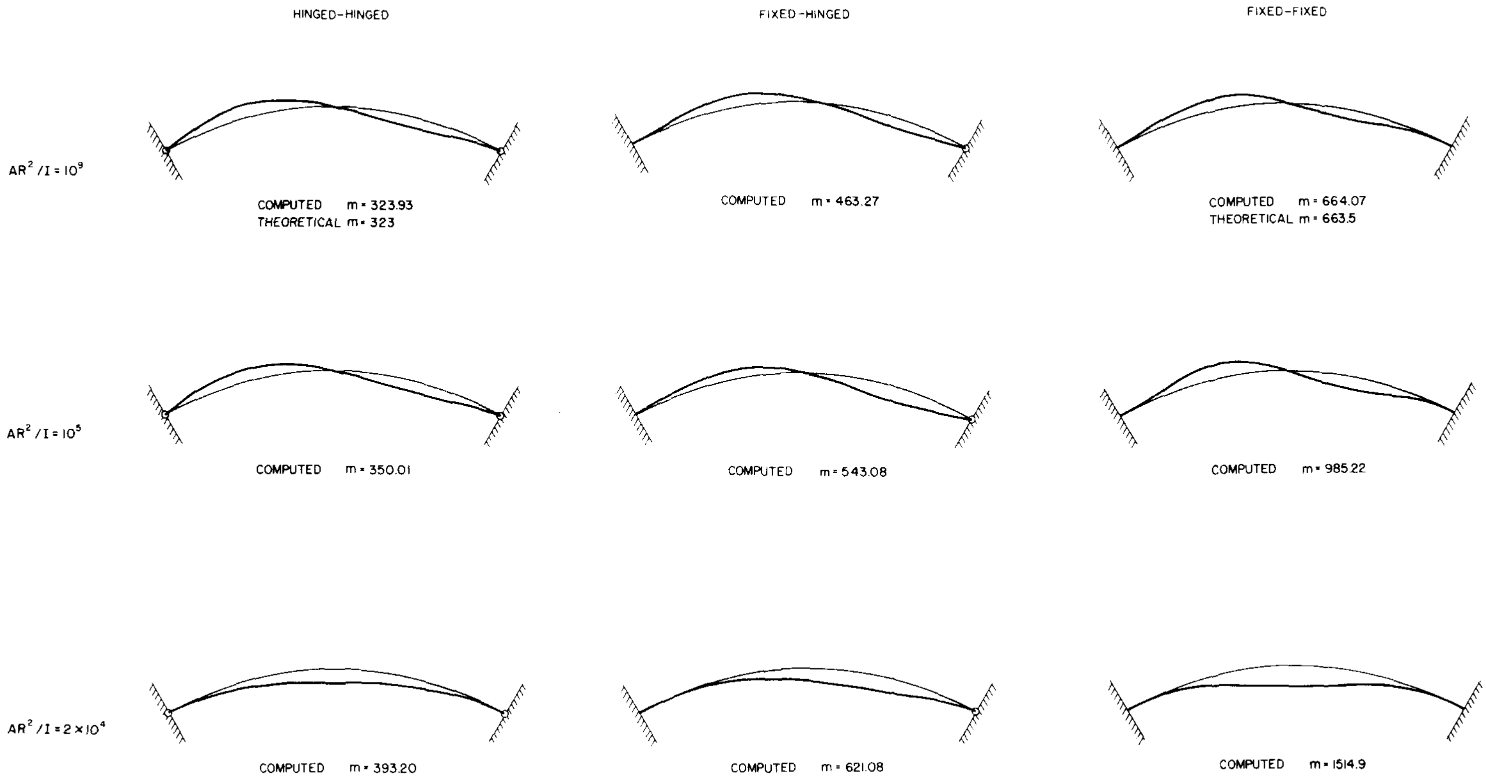
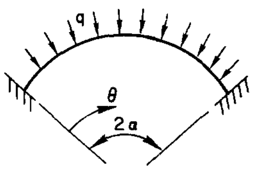


FIG. 4. Buckling loads and buckled shapes for shallow circular arches ($\alpha = 10°$), $q_{cr} = m\frac{EI}{R^3}$ (theoretical values from [1] pp. 297–301).

Table 2. Coefficients k for buckling of uniform columns $P_{cr} = k\frac{\pi^2 EI}{L^2}$

Mode number	Hinged–hinged			Fixed–hinged			Fixed–fixed			Fixed–free		
	Present results	Theory	% diff.	Present results	Theory	% diff.	Present results	Theory	% diff.	Present results	Theory	% diff.
1	1·0000	1	~0	2·0458	2·0457	~0	4·0001	4	~0	0·25002	0·25	0·01
2	4·0001	4	~0	6·0470	6·0469	~0	8·1837	8·1828	0·01	2·2518	2·25	0·08
3	9·0006	9	0·01	12·049	12·047	0·02	16·006	16	0·04	6·2633	6·25	0·21
4	16·004	16	0·03	20·057	20·047	0·05	24·206	24·188	0·07	12·307	12·25	0·47
5	25·015	25	0·06	30·083	30·047	0·12	36·090	36	0·25	20·395	20·25	0·72
6	36·047	36	0·13	42·154	42·047	0·25	48·373	48·188	0·38	30·653	30·25	1·33
7	49·126	49	0·26	56·333	56·047	0·51	64·738	64	1·15	42·954	42·25	1·67
8	64·300	64	0·47	72·767	72·047	1·00	81·384	80·189	1·49	58·103	56·25	3·29
9	81·653	81	0·81	91·908	90·047	2·07	113·34	100	13·3	74·871	72·25	3·63
10	101·32	100	1·32	125·97	110·05	14·5	135·94	120·19	13·1	111·44	90·25	23·5

TABLE 3. COEFFICIENTS m FOR BUCKLING OF FIXED CIRCULAR ARCHES $q_{cr} = \frac{mEI}{R^3}$ WITH $\frac{AR^2}{I} = 10^7$

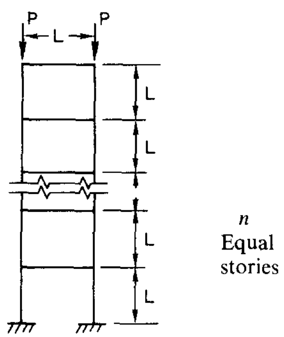


Arch angle 2α °	Theoretical results [11, p. 848]	98 Element solution with nine generalized loads	Four element direct solution
60	~74·9†	74·96	78·05
120	19·4	19·59	21·36
180	9·0	9·003	10·60

† See [14].

(vii) *Multistory frame.* The single bay frame analyzed was composed of from three to fifteen stories, each of height L and bay width L. Equal section properties were assumed for all columns and girders in the frame. Independent loads consisted of transverse forces on the columns of the frame, and included five symmetric loads $q = \sin n\pi x/H$, and five antisymmetric loads $q_n = \sin(2n-1)\pi x/2H$, where H is the total height of the frame. Results for the lowest buckling load are given in Table 4. They are in excellent agreement with exact values which were computed using a direct eigenvalue solution. The efficiency of the present method can be seen by comparing the respective computation times. The direct solution for a 15 story frame took approximately fifteen times longer than the present method.

TABLE 4. COEFFICIENTS m FOR BUCKLING OF MULTISTORY FRAMES $P_{cr} = \frac{mEI}{L^2}$ WITH $\frac{AL^2}{I} = 10^4$



Number of stories n	Present results	Direct eigenvalue solution
3	4·546	4·542
5	4·127	4·124
10	3·786	3·775
15	3·505	3·487

5. CONCLUDING REMARKS

It can be seen from the examples that the present method is an effective and practical means of predicting critical loads. The method is related to the Rayleigh–Ritz procedure and the present results are upper bounds to the exact solutions. It has been shown that the accuracy of the upper bounds for the lowest eigenvalue can be improved with an increased number of admissible coordinate functions (see Chen [12], for example). If the accuracy of the present solution needs to be verified, then the Vianello or Stodola method can be applied, using the present results as a point of departure. As demonstrated by the examples, the accuracy of the present results is of sufficient quality that if iteration is necessary, only a minimal number of cycles should suffice for convergence. The use of a static load pattern in an eigenvalue problem can be traced back to Morley [13]. The present method can therefore be considered as an extension of this concept for more than one generalized coordinate.

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Абстракт—Дается метод анализа для статической теории бифуркации, используя сокраженную систему обобщенных координат. Для идеализации конструкции используются конечные элементы модели перемещений/. Преимуществом предлягаемого метода является большое сокращение размера задачи на собственные значения, которую можно определить боз отказывания способности использования большого числа степеней свободы для математической модели. С помощью численных примеров иллюстрируются эффективность и точность метода для разных типов конструкций. Несмотря на это, что в настоящей работе даются только примеры, касающиеся одномерных элементов, процесс, очевидно, можно приметить для упругих конструкций, таких как пластинки и оболочки.